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★**An introduction to infinite-dimensional dynamical systems—geometric theory.**

With an appendix by Krzysztof P. Rybakowski.

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From the introduction: "The purpose of these notes is to outline an approach to the development of a theory of dynamical systems in infinite dimensions which is analogous to the theory of finite dimensions. The first problem is to find a class for which there is some hope of classification and yet general enough to include some interesting applications. Throughout the notes, the discussion centers around retarded functional-differential equations although the techniques and several of the results apply to more general situations; in particular, to neutral functional-differential equations, parabolic partial differential equations and some other types of partial differential equations.

"Let X,Y,Z be Banach spaces (sometimes Banach manifolds) and let $\mathfrak{X}^r=C^r(Y,Z),\ r\geq 1$, be the set of functions from Y to Z which are bounded and uniformly continuous together with their derivatives up through order r. We impose the usual topology on $\mathfrak{X}^r$. For each $f\in\mathfrak{X}^r$, let $T_f(t)\colon X\to X,\ t\geq 0$, be a strongly continuous semigroup of transformations on X. For each $x\in X$, we suppose $T_f(t)x$ is defined for $t\geq 0$ and is C^r in x. Let $A(f)=\{x\in X\colon T_f(t)x$ is defined and bounded for $t\in(-\infty,\infty)\}$. The set $A(f)$ contains much of the interesting information about the semigroup $T_f(t)$. The basic problem is to discuss detailed properties of the set $A(f)$ and to determine how $A(f)$ and the structure of the flow on $A(f)$ change with f.

"If $A(f)$ is not compact, very little is known at this time. It becomes important therefore to isolate a class of semigroups for which $A(f)$ is compact. If $T_f(t)$ is an α-contraction for $t>0$ and $T_f(t)$ is compact dissipative, then it can be proved that $A(f)$ is compact. We give two examples of semigroups which can be used as models to illustrate several of the ideas. Suppose $u\in\mathbf{R}^k,\ x\in\mathbf{R}^n$, Ω is a bounded, open set in $\mathbf{R}^n$ with smooth boundary, D is a $k\times k$ constant diagonal, positive matrix, Δ is the Laplacian operator, and consider the equation $u_t-D\Delta u=f(x,u,\operatorname{grad}u)$ in Ω, $u=0$ on $\partial\Omega$. Other boundary conditions could also be used. Let $W=W_0^{1,2}(\Omega)\cap W^{2,2}(\Omega)$ be the domain of $-\Delta$ and let $X=W^\alpha, 0<\alpha<1$, be the domain of the fractional power $(-\Delta)^\alpha$ of $-\Delta$ with the graph norm. Under appropriate conditions, this equation generates a strongly continuous semigroup $T_f(t)$ on X which is compact for $t>0$. In this case $\mathcal{X}^r = C^r(\Omega\times\mathbf{R}^k\times\mathbf{R}^{kn},\mathbf{R}^k)$. If f is independent of x, then $\mathcal{X}^r = C^r(\mathbf{R}^k\times\mathbf{R}^{kn},\mathbf{R}^k)$. If f depends only on u, then $\mathcal{X}^r = C^r(\mathbf{R}^k,\mathbf{R}^k)$. In each of these cases, the theory will be different.

"As another example, suppose $r>0$, $C=C([-r,0],\mathbf{R}^n)$, $\mathcal{X}^r = C^r(C,\mathbf{R}^n), r\geq 1$, and consider the RFDE, $\dot{x}(t)=f(x_t)$ where, for each fixed t, x_t designates the restriction of a function x as $x_t(\theta)=x(t+\theta)$, $-r\leq\theta\leq 0$. For any $\varphi\in C$, let $x(\varphi)(t)$, $t\geq 0$, designate the solution with $x_0(\varphi)=\varphi$ and define $T_f(t)\varphi=x_t(\varphi)$. If this function is defined for $t\geq 0$, then $T_f(t)\colon C\to C$ is a strongly continuous semigroup and $T_f(t)$ is completely continuous for $t\geq r$ if it takes bounded sets to bounded sets.

"For differential-difference equations $\dot{x}(t)=f(x(t),x(t-r))$ or $\dot{x}(t)=f(x(t-r))$, the space $\mathcal{X}^r$ is, respectively, $C^r(\mathbf{R}^n\times\mathbf{R}^n,\mathbf{R}^n)$ or $C^r(\mathbf{R}^n,\mathbf{R}^n)$.

"The abstract dynamical systems above also include some neutral functional-differential equations and other classes of partial differential equations.

"Some basic questions that should be discussed are the following: Q.1. Is $T_f(t)$ one-to-one on $A(f)$ generically in f? Q.2. If f is A-stable, is $T_f(t)$ one-to-one on $A(f)$? Q.3. When is $A(f)$ a manifold or a finite union of manifolds? Q.4. Can $A(f)$ be imbedded in a finite-dimensional manifold generically in f? Q.5. For each $x\in A(f)$, is $T_f(t)x$ continuously differentiable in $t\in\mathbf{R}$? Q.6. Are Kupka-Smale semigroups generic? Q.7. Are Morse-Smale systems open and A-stable? Notice that all questions are posed for $A(f)$.

"In these notes, we discuss in detail how one can obtain a geometric theory for retarded functional-differential equations and we attempt to answer some of the questions above. Throughout the notes, we point out when the techniques and results are applicable to the more general abstract framework. We have attempted to give a unified exposition of some of the fundamental results in this subject, always making the presentation as self-contained as possible. Some parts of the notes are also devoted to speculations on the directions for future research."